All-purpose minimal sufficient networks in the threshold game

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Abstract:
This paper considers a multi-player stag hunt where players differ in their degree of conservatism, i.e. in the threshold of players that need to act along with them before they see benefits in collective action. Additionally, any player is either available for action or not. Minimal sufficient networks, which depending on their thresholds allow players to achieve just enough interactive knowledge about each other's availability to act, take the form of hierarchies of cliques (Chwe, 2000). We show that any typical threshold game has a plethora of such networks, so that players seem to face a large degree of strategic uncertainty over which network to use. The plethora of networks includes cases where the structure of the network infects players into acting more conservatively than is reflected in their thresholds. An extreme case of this is the core-periphery network, where each player acts as conservatively as the most conservative player that can exist in the population. Because of this feature, the core-periphery network is minimal sufficient for all possible populations. Players can thus solve the strategic uncertainty arising from the multiplicity of minimal sufficient networks by using the all-purpose core-periphery network.

The class of collective action problems known as *stag hunt* games (for an overview, see Skyrms, 2004) are characterized by an "I'll go if you go" mechanism: if you innovate (revolt against the government, use a new technological standard,…), so will I. Contrary to what is the case in the prisoner's dilemma, both inaction and collective action are Nash equilibria. But given the large cost of acting alone, even the slightest doubt that others do not act can induce the individual player not to act, for this reason it is also referred to as the trust dilemma. In the two-player stag hunt, each player only acts when somehow receiving assurance that the other player acts – for this reason this game is also known as the assurance game (Sen, 1967). In multi-player stag hunts (Carlsson and Van Damme, 1993), players may moreover differ according to the number of people they want to act along with them before they find it worth to act, and may thus also differ according to the level of assurance that they require. Granovetter (1978) refers to a player's *threshold* as the critical number of players that need to act for benefits of collective action to arise for him. For this reason, this type of collective action has also been referred to as a *threshold game* (Chwe, 1999, 2000). A radical, low-threshold player may act as soon as receiving assurance from even a single other player. A conservative, high-threshold player on the contrary will need assurance from many others. The player population may span all thresholds in between these extremes.

In terms of the network literature (Jackson and Wolinsky, 1995; Bala and Goyal, 2000), a message by player i to player j by which player j finds out player i's willingness to act can be seen as a link from player i to player j. The question arises then: what sort of network structure needs to be established between the players for them to achieve collective action? Granovetter (1978) suggests a bandwagon network, in which players order themselves according to their thresholds. In the simplest case, there is one player of each threshold, including a player who does not require assurance from anyone, namely a threshold-1 player. As this player is still better off the more other players act, the threshold-1 player assures the threshold-2 player of his willingness to act, after which the threshold-3 player is told about the willingness to act of the two first players, etc. Yet, as pointed out by Granovetter, such a bandwagon does not assure collective action for all populations. In fact, in an only slightly different population with the threshold-1 player turned into a threshold-2 player and all other players as before, any attempt to use the same bandwagon to achieve collective action leads to complete inaction. The former threshold-1 player no longer acts, because she would now like information from at least one other player. As she does not get such information, she cannot convince the threshold-2 player to act. Etc. Granovetter's reason for providing this example is to show that one cannot simply talk of some type of group intent, where one considers a population as an entity, and where two very similar populations would then be expected to act in the same manner. An alternative interpretation is that it would then be nice to have a general-purpose type of communication network, that works for any population. Unfortunately, Granovetter's example suggests a different type of communication network may have to be designed for each individual population, where even two only slightly different populations may require a different communication network.

The purpose of this paper is to show that a general-purpose network, which works for any population, does exist across different threshold games. We show this by treating a modified version of Chwe's (2000) formal model of the threshold game. Just as in Chwe's original model, we obtain that any network that makes all players act (i.e. is *sufficient*) and has no redundant links (i.e. is *minimal*) takes the form of a *hierarchy of cliques*. A clique is a subset of players who all talk to each other. Any minimal sufficient network partitions the players in cliques, and cliques talk to one another only in one direction.[1] A chain of cliques is thus obtained, consisting of one or more leading cliques of radicals, who communicate with cliques of somewhat less radical players, who again communicate with still less radical cliques, etc. While Chwe's propositions do not imply such restrictions, in the examples of minimal sufficient networks that he provides, all players are connected in a single network, containing multi-player cliques. Each individual clique is homogeneous, and contains all players of a certain threshold. Finally, players are ordered in the hierarchy according to their thresholds, in that higher threshold players are systematically at a lower rank in the hierarchy.

Our first contribution is to show that for any population, in line with Chwe's results but contrary to the examples that he provides, typically a plethora of minimal sufficient networks exists (this result is obtained both in Chwe's original model, and in the modified version of it presented in this paper). These may not only include Granovetter-like bandwagon networks consisting only of one-player cliques, but also networks containing only one-player cliques (thus many hierarchical ranks), networks containing as few as two hierarchical ranks (one leader clique, and follower cliques depending on it), networks containing multi-player cliques consisting of players with different thresholds (heterogeneous cliques), networks where inherently more radical players are at a lower hierarchical rank than more conservative players, and networks consisting of several isolated components.

The surprising result here is the existence of minimal sufficient networks consisting of heterogeneous cliques and/or networks putting more radical players at a lower hierarchical rank. These two effects exist for the same reason, namely that one needs to make a difference between a player's own exogenously given threshold, and the endogenous threshold enforced on him by his social position in the network. Consider a three-player game where player 1 has threshold 2, and where players 2 and 3 have thresholds 3. At first sight, it would seem that is suffices that player 1 finds out that player 2 is (in principle) willing to act. Yet, this does not guarantee that player 2 will effectively act, as player 2 himself needs to find out first that player 3 is willing to act. Thus, player 1 will only be willing to act if player 2 and player 3 are linked to one another. Moreover, this does not suffice, as player 2 could also find out that player 3 is not willing to act. Thus player 1 should also find out whether player 3 is willing to act. While player 1's exogenous threshold is 2, the structure of the social network has modified his behavioral threshold into 3. Similarly, consider a four-player game with one threshold-2 player and three threshold-3 players. Let the three threshold-3 players form a leading clique (meaning that they do not receive information from anyone outside of the clique). The threshold-2 player will not be content with finding out that one of the threshold-3 players is in principle willing to act.

[1] More correctly, in Chwe cliques can also overlap. Our simplifying assumptions exclude such cases.

Knowing that this threshold-3 player only acts when he finds out that the two players in his clique are willing, the threshold-2 player will want to find out whether these two other players are willing as well. Effectively, the threshold-2 player behaves as a threshold-4 player, and thus effectively behaves more conservatively than the inherently more conservative players in the leading clique. This result may be seen as formalizing the sociological idea of *embeddedness* (Granovetter, 1985), saying that one's position in a network has an influence on one's behaviour.

On the negative side, the existence of a plethora of minimal sufficient networks shows that players face considerable strategic uncertainty. Which of these many networks should they coordinate on using? Yet, exactly because of the multiplicity of networks in each individual game, perhaps one type or at least a few types of minimal sufficient networks exist that work for all populations. In this case, such networks are bound to be focal, and the players would be able to resolve the problem of strategic uncertainty. Examples of such possible network types we investigate are networks with everyone in one-player cliques; networks with all players with the same threshold in one and the same clique; and core-periphery networks with a large number of players in a leading clique, with all other players around it in one-player follower cliques. While some network types work for a relatively large number of populations, unfortunately we are not able to find a network formation rule that works for all populations, thus generalizing Granovetter's intuition that minimal sufficient networks are population-specific.

Our initial result of non-existence of generally applicable network formation rules is partly due to the requirement of *minimal* sufficiency for any individual population. Trivially, eliminating the requirement of non-redundancy, players can always achieve collective action if they use the complete network in any population. We show that players can use less links than in the complete network, and thus still approach minimality, if they use a core-periphery network. Denoting as $t_{\max}$ the threshold of the most conservative player(s) in the population, the core-periphery network consists of a leading clique of *any* subset of $t_{\max}$ players from the population, and the rest of the players in one-player cliques depending on it. Put otherwise, it does not matter at all where players are positioned in the network. A core-periphery network thus works even for players who do not have any information about the population other than the number of players and the threshold of the most conservative player(s). The core-periphery network is still *minimal* sufficient for such uninformed players, in that within each set of populations with the same number of players and the same maximal threshold, a subset of populations exists for which the core-periphery network is minimal.

The use of a core-periphery network to solve collective action problems has some intuitive appeal. First, a leading clique is formed of a size large enough to bring a consensus among even the most conservative players about the desirability to act. Such a leading clique may be considered as a committee containing a representative sample of the population. Indeed, on average, the players in the committee are distributed in the same manner as the population. Second, all remaining players receive information that the committee has reached a consensus on when and where to act. Given that this committee is large enough to make even the most conservative player act, a message from this committee suffices to any player excluded from the committee. Players excluded from the committee do not additionally need to talk to each other.

The paper is structured as follows. Section 2 treats a modified version of Chwe's (2000) model. Section 3 treats an example of a threshold game, and shows that on top of the type of examples of networks suggested by Chwe and Granovetter, a plethora of other networks may be minimal sufficient, thus potentially leading to a large amount of strategic uncertainty. Section 4 investigates several types of networks, with the purpose of investigating whether there is a type of network that is minimal sufficient for all types of populations. Section 5 treats a modified concept of minimal sufficiency, where minimality now means that a network is minimal for at least one population in the class of populations with the same number of players and the same maximal threshold. It is shown that the core-periphery architecture is the unique network formation rule that allows players to achieve a minimal (in the new sense) sufficient network for each class of populations, as characterized by the number of players, and the maximal threshold). The paper ends with a conclusion in Section 6.

2. Game-theoretical threshold model and minimal sufficient networks

Let us attempt to construct in the simplest manner possible a game-theoretic model of sociological threshold models. As collective action should be an equilibrium, we need a stag hunt game rather than a prisoner's dilemma. In order for players to have different thresholds, this must somehow be reflected in their payoffs. We thus obtain a multi-player stag hunt game with heterogeneous players. The game is played by a finite set of players $N = \{1,2,...,n\}$. Each player has a *threshold* t_i, with $2 \le t_i \le t_{\max}$. $t_{\max}$ is the highest threshold in the game, where we assume that $t_{\max} \le n$. Each player $i \in N$ simultaneously chooses an action $a_i \in \{r, s\}$, where *r* is the risky action, which we will refer to as "action", and *s* is the safe action, which we will refer to as "inaction". A player who takes action *s* always obtains payoff zero, whatever his threshold. When taking action *r*, the payoff of a player *i* with threshold t_i depends on R_{-i}, i.e. the number of other players than *i* who take action *r*. When $R_{-i} < (t_i - 1)$, player *i* obtains payoff $-L$ when doing *r*, where L is a large loss. When $R_{-i} \ge (t_i - 1)$, player *i* obtains payoff $M(R_{-i}) > 0$ when doing *r*, with $M'(R_{-i}) > 0$, meaning that player *i* incurs a positive payoff as long as action with together with t_i players or more, where this payoff is then larger the more players he acts together with.[2] All aspects of the game, including the players' thresholds, are common knowledge. Common knowledge of thresholds is in contrast to Chwe (2000). We assume such common knowledge because we are interested in endogenous network formation in a given population facing a collective action problem, and not in the properties of exogenously given networks in which collective action is likely.[3]

Any such stag hunt game has at least two Nash equilibria. Given that $t_{\max} \le n$, an equilibrium exists where everyone acts. Given that nobody acts when nobody else acts, there is also an equilibrium where nobody acts. Let us now look at two examples. In example 1, the population consist of thresholds $(2,2,3,4,5,6,7,8,9,10)$, in example 2, the population consist of thresholds $(2,3,4,5,6,7,8,9,10,10)$. Both examples have exactly two strict Nash equilibria, namely the ones identified above. Given the loss when acting with less players than one's threshold, in both examples the efficient equilibrium is risk dominated (Harsanyi and Selten, 1988). From a game-theoretic perspective, the two examples would thus seem very similar. Yet, from the perspective of the literature on threshold models (Granovetter, 1978; for an overview of recent literature, see Vanderschraaf, 2008), the two examples are very different. In example 1, if the two threshold-2 players both expect each other to act, this induces them to act. As soon as they are known to act, this suffices to create a chain reaction that makes everyone act: their action will trigger action by the threshold-3 player, who will again trigger action by the threshold-4 player, etc. In example 2, however, there is no subset of lower threshold players that would act without knowing whether the rest of the population acts. This is because, as can be checked, the maximal threshold in any subset is always larger than the number of players in the subset. The

[2] This is contrast to stag hunt models of partner choice, where players are assumed to look for a sufficient number of cooperative partners in a population (Corbae and Duffy, 2007).

[3] For an overview of endogenous network formation in economics, see Jackson (2003). In sociology, see Lazer (2000).

only manner for the players to achieve collective action in example 2 is for all of them to agree together to act.

Why are the two examples very similar in the game-theoretic model[4] and very different in the threshold model? In the threshold model, the assumption is that the threshold-2 players somehow assure one another that they will act. Either they now communicate their intention to act to the threshold-3 player, or the threshold-3 player simply observes them acting. The threshold-3 player now acts as well. Again, he communicates this to the threshold-4 player, or this player observes the threshold-4 player acting. Etc. The structure of the communication network, or the order in which the players move, plays an essential role in this model. Yet, in the game-theoretic model treated above, players make their decisions to act simultaneously. Moreover, communication does not make any difference in this game-theoretic model treated. To see why, suppose that players can communicate in a shared language to one another their intention to act, and suppose that the threshold-2 players in example 1, previously to playing, communicate to one another their intention to act. The problem with this is that such a message is not credible, as one has a weak incentive to send such a message even when one is not planning to act (Aumann, 1990). Yet, experiments have shown that pre-play communication leads to play of the efficient equilibrium in stag-hunt games (Cooper et. al., 1992).

In game-theoretic terms, players communicating their intentions to act only makes sense if there are two types of players, namely players who do not intend to act, and players who possibly intend to act.[5] We could think here of players who are available for action, and players who are not available for action. Thus, in order to bring our model to being closer to being a game-theoretic account of the threshold model, we need to turn our game into one with asymmetric information. We assume that each player i, fully independently from his threshold t_i, is with probability $(1 - \varepsilon)$ in state w (available for action), and with probability ε in state x (not available for action). In state w, the player has the payoffs described above. In state x, the player again always obtains payoff 0 when not acting, but now always obtains payoff $-L$ when acting. When each player is in state w, we have the stag hunt game treated above. The presence of the state x formalizes the individual player's doubts about whether the other player will actually act. With probability ε, the individual player goes crazy and perceives his payoffs in such a way that inaction is a dominant strategy.

Summarizing, we obtain the following stag hunt with asymmetric information and heterogeneous players. At stage 1, Nature determines for each player i a threshold t_i, and a state w (probability $(1 - \varepsilon)$) or x (probability ε). The state occur with the same probability, whatever a player's threshold. The thresholds chosen by Nature are common knowledge, the state is not. At stage 2, communication takes place between players (see below). At stage 3, each player simultaneously decides to do action s or action r. At stage 4, each player obtains his payoff, as specified above.

As long as L is large, a necessary condition for a player with threshold t_i to act involves receiving messages from at least $(t_i - 1)$ other players signaling that they are in state w. It should be noted that an equilibrium without communication where no player acts continues to

[4] It should be noted that the examples are still quite different if one induces learning dynamics. Suppose that the games are played repeatedly, and that players' strategies are subject to noise. Then in example 1, starting from the inefficient equilibrium, if noise happens to make both threshold-2 players act, then they will both find out that this makes them better off, and will stick to this new strategy. As soon as this has happened, the threshold-3, threshold-4, etc. player will find out that it is better for them to act. In example 2, however, the efficient equilibrium is only learned if noise happens to make all players play take action. But these learning dynamics then simply replicate the communication process that players could achieve by establishing a communication network.

[5] A similar game where asymmetric information is added to a stag hunt game in order to make communication relevant is the electronic mail game (Rubinstein, 1989). The focus there is on noisy communication, and on how this leads players to require a large number of confirmations and reconfirmations from one another.

exist. Thus, our model does not solve the issue of equilibrium selection; it only ensures that, *if* the efficient equilibrium is played, this is conditional on communication having taken place. For simplicity, we do not model this communication strategically, but instead study the properties of communication networks that just still allow players to achieve collective action. These networks may be seen as assurance networks, establishing trust among the players that a sufficient number of players is in state w and that their individual thresholds will be achieved. Concretely, we say that player i has a link with player j if $g_{i,j} = 1$, and that player i has no link with player j if $g_{i,j} = 0$. We consider the case where links correspond to one-way communication: $g_{i,j} = 1$ enables agent j to access i's information on whether i is in state w or x, but not vice-versa. Graphically, we denote $g_{i,j} = 1$ as an arrow pointing from player i to player j: j is able to observe the dimension q_i of player i, so that player i's type is communicated to j. However, following Chwe, we assume that link $g_{i,j} = 1$ link $g_{i,j} = 1$ does not give player j access to the information that player i may have about any information that player i may have about the type q_h of a player h through a link $g_{h,i} = 1$. In terms of the network literature, this may be seen as an extreme case of information decay (Bala and Goyal, 2000): as the distance in the network increases, the value of information decreases – which takes place here in an extreme way. Another way to formulate this assumption is to say that links *cannot aggregate information*; any player can only communicate whether his own state is w or x, not what he finds out about other players' states, or what other players have found out about other players' states. A typical set of all player i's links and non-links is denoted as g_i, where $g_i = \left(g_{i,1},...,g_{i,i-1},g_{i,i+1},...,g_{i,n}\right)$. Define as a *network*, g, a set of sets g_i for each player i, thus $g = \left(g_1,...,g_n\right)$. We focus on networks that allow *all* players to act. If the population is such that there are also equilibria where only a few players act, this is justified in terms of the preferences of any player in state w, who is better off the more other players act.

Definition 1.
Define as a *sufficient network*, any network that allows for an equilibrium where all players act when all players are in state w.

But a trivial sufficient network is then simply the complete network, where all players observe each other's types q_i. While we do not model network formation as strategic, we can still impose plausible restrictions on networks, which are likely to arise upon strategic network formation. Given the large L, no player will want to run risk, and will thus always require at least the minimal number of messages assuring that his threshold is achieved. By increasing the number of messages attended to above this minimal number, the player can increase the probability of collective action. Yet, the more links a player attends to, the more costs he will incur. For simplicity, we assume that these costs are always so high that the player prefers to pay attention to the minimal number of messages assuring that he achieves his threshold. For this reason, we focus on networks where each player considers each message as crucial. As soon as he does not receive a crucial message, he does not act and obtains payoff zero.

Definition 2
Define as a *minimal sufficient network* g (henceforth *msn*), any sufficient network with the following property. Consider *any* subset of players N', and denote by $g_{N'} \subseteq g$ the set of links received by these players. Let it be the case that as soon as any subset of links in $g_{N'}$ is deleted, in the newly obtained communication network, no Nash equilibrium exists where the players in set X, with $X = N'$ all act.

Our concept of minimality differs from the one of Chwe (2000), in that in Chwe, X is equal to N as a whole rather than to N'. Chwe's concept of minimality is that any sufficient network g is minimal sufficient as long as there does not exist a network $g' \subset g$ that *also* allows for an equilibrium with collective action of all players. Chwe's concept thus sees a network as unstable if deletion of messages can lead to a new network that is also sufficient. In our concept of minimality, which lies closer to the network literature, any sufficient network g is minimal sufficient as long as there does not exist a network $g' \subset g$ that allows the players in g' receiving less messages than in g to still act, where crucially it need not longer be the case that the players not in g' still act. Thus, in our concept of minimality, any subset of players N' should consider the messages currently received as crucial. The effect of deletion of messages from g on the decision to act of players in the set $N \backslash N'$ is not taken into account. Given our modified concept of minimality, and given that information is local, it is easy to show that each *msn* takes the form of a hierarchy of cliques. We do this by introducing some more definitions, and by proving some intermediate lemmata.

Definition 3
We say that there is a path from j to i in g if there exist agents $j_1,\ldots,j_m$ such that $g_{i,j_1} = g_{j_1,j_2} = \ldots = g_{j_m,j} = 1$.

Lemma 1. Under local knowledge (A2), if a path $g_{i,j_1} = g_{j_1,j_2} = \ldots g_{j_m,j} = 1$ exists in a *msn*, given (A2), it must also the case in this *msn* that $g_{i,j} = g_{j_1,j} = g_{j_2,j} = \ldots g_{j_m,j} = 1$.
Proof: Given that information is local, player j only observes the threshold of player j_m. At the same time, as the network structure is common knowledge, player j knows that players j_1, j_2, …,j_m only act when each, on this path, finding out the right thresholds. Given the high risk of acting with less players than the threshold, player j needs direct information from all players on the path.

Definition 4.
Define as a *cycle* any path $g_{i,j_1} = g_{j_1,j_2} = \ldots = g_{j_m,i} = 1$.

Definition 5.
Define as a clique a set of players $j_1, j_2, \ldots, j_m$ such that for all j_m, j_n in this set of players, we have $g_{j_m,j_n} = g_{j_n,j_m} = 1$.

Corollary 1. If a *msn* contains a cycle, then all the players in this cycle are in one and the same clique.
Proof: This follows from the fact that there is path between any two players in a cycle, and from Lemma 1.

Corollary 1 is the natural consequence of the assumption that information is local. If it is crucial to player j to receive a message from player i, then player j only acts when knowing that player i will act. But, knowing the network structure, in which player i only acts when receiving a certain number of messages, and given that information is local, player j will only act when also receiving all messages that player i wants to receive. Moreover, as i reasons in a similar manner about messages that the players sending to him receive, player j also wants to receive these messages. Etc. Corollary 2 now shows that this implies that cliques in a *msn* cannot overlap. This is contrary to Chwe (2000), whose different concept of minimality does allow for overlapping cliques. An example is treated after the proof.

Corollary 2. *Msn*s can only contain two cliques that share players if these players are together in a single clique.
Proof: When two cliques share players, this means that there is a cycle involving all players in these two cliques. But then, by Corollary 1, they must be in a single clique.

We further show that if a player i in clique A talks to a player j in a separate clique B, then no player h in clique B can talk to any player k in clique A, and that if one player in clique A talks to one player in clique B, then all players in clique A must talk to all players in clique B.

Lemma 2. Consider two separate cliques in a *msn*, denoted as clique A and clique B. If a player i from clique A talks to a player j from clique B, then no player k form clique B can talk to any player h from clique A.
Proof: Suppose that a player i from clique A talks to a player j from clique B, and that a player k form clique B talks to a player h from clique A. Given that in each clique, all players talk to each other, there is then a cycle encompassing all players in A and B. But then all these players should talk to each other.

Lemma 3. Consider two separate cliques in a *msn*, denoted as clique A and clique B. If a player i from clique A talks to a player j from clique B, then *all* players from clique A should talk to all players from clique B. In short, we then say that clique A talks to clique B.
Proof: This follows directly from the fact that the presence of one link implies that there is a path between any pair of players divided over the cliques.

THEOREM 1. Any *msn* for an individual population state takes the form of a partition of the players in cliques, where any two cliques may talk to one another in only one direction, and where there are no cycles among cliques.
Proof: By Corollary 1, any players contained in a cycle of links must be in on and the same clique. By Lemmata 2 and 3, any two cliques can only talk to one another in one direction, where one clique talking to another means that all players in the former clique talk to all players in the latter clique. These "talking to" relations between cliques may not form cycles, since otherwise the players in these cycles cannot be in separate cliques.

Since by Theorem I there cannot be cycles among the cliques, any *msn* must contain at least one *leading clique*, characterized by the fact that players in the clique do not receive any messages from players outside of the clique, and at least one *end clique*, characterized by the fact that players in the clique do not send any messages to players outside of the clique. Moreover, any msn can be seen as a set of *chains of cliques*, where a chain of cliques is any path in a *msn* between a leading clique and an end clique in the network (where a link from clique A to clique B on such a path means that all players in clique A talk to all players in clique B). In every chain of cliques, all links between cliques point from the leading clique to the end clique. We add the following definition.

Definition 6. Define as a player's *rank* in a clique chain the number of cliques from which his clique receives information, minus one.

Thus, the players in the leading clique of a clique chain have rank 1 (the highest rank) in this clique chain, the players in the clique receiving messages from the leading clique have rank 2 (the one-but highest rank), etc. Following Chwe, cliques can be interpreted as *social roles* (instigators, immediate followers of the instigators, etc.), and any *msn* can be interpreted as a *hierarchy of social roles*.

A first intuition for why *msn*s must take the form of a hierarchy of cliques is provided for the simple case in Figure 1, in which players I, II, III and IV all have threshold 3, and all know each other's thresholds. An arrow denotes message(s) sent in one direction, whereas a line denotes messages sent in one direction. In Figure 1a, suppose that player I receives a message from players II and III, and sends a message to both these players. Suppose now that I finds out that II and III are in state *w*. This does not mean that II and III will act; given their thresholds, they will only act if they hear from two other players that they are in state *w*. Thus, I will only act if II and III also talk to one another. We then obtain a clique of three players as indicated in Figure 1b, where we always draw a circle around the players in who are in the same clique. How about IV? Does it suffice that e.g. II and III tell him that they are in state *w* (indicated by arrows in Figure 1c)? It does not, because IV knows that, given the network structure, II and III only act if they find out from I that she is in state *w*. So, IV must receive a message from each player in the three-player clique, as indicated in Figure 1d. Effectively, we thus have a three-player clique talking to a one player clique. Figure 2 gives a simpler representation of the *msn* in Figure 1d. A line within a clique now denotes that people talk to each other; an arrow from one clique to an other clique denotes that everyone in the former clique talks to everyone in the latter clique.

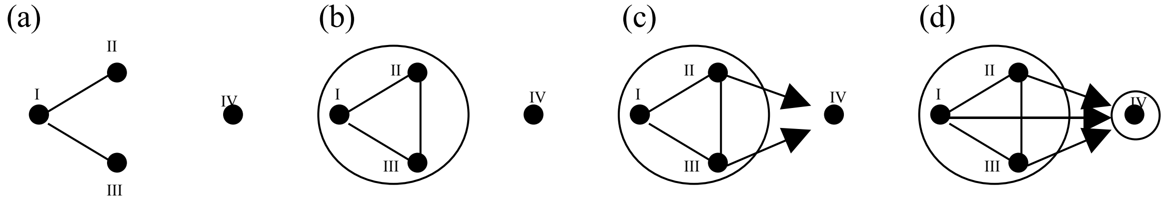


Figure 1 Clique formation

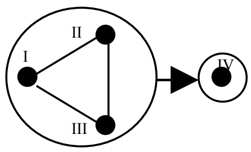


Figure 2 Notation for communication between cliques

We next treat two networks for the same example that are sufficient, but not minimal. Figure 3 represents the complete network. Suppose that players II, III and IV require three instead of two messages that other players are in state *w* before they are willing to act. Then it is a best response for player I to also require three messages. Given the symmetry of the example, this means that requiring three messages is then a best response to every player. Yet, the entire set of players should jointly realize that they can still coordinate on collective action if IV does not send messages to I, II and III, and if I, II and III content themselves with messages among themselves that they are all in state *w*. This argument coincides with minimality as defined in Chwe (2000).

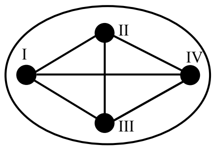


Figure 3 Complete network: sufficient, but not minimal

The network in Figure 4 consists of two overlapping cliques. I and IV now each require messages from both II and III that they are in state *w*, but do not require messages from one another. II and III require such a message from one another, and additionally a message from either I or IV (they can't require a message from both I and IV, since otherwise I and IV will also want to know of one another). In Chwe (2000), this network is minimal, since one cannot delete links and move to a new sufficient network that allows collective action. In our modified version of minimality, this network is non-minimal, as II and III would still act without a message from IV, and thus can decide not to pay attention to this message in the first place. However, given that IV sees that II and III do not pay attention to his messages anymore, IV is no longer assured that II and III act, as he does not know whether II and III found out that I is in state *w*. It follows that IV does not act. Summarizing, Chwe calls a network non-minimal if you can make a smaller subnetwork that is still sufficient. We call a network non-minimal if one or more players do not have any incentive to pay attention to all signals, and this independent of the fact whether we still have a sufficient network after these players stop paying attention to messages.

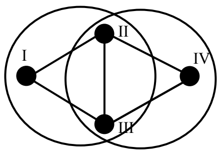


Figure 4 Overlapping cliques: sufficient, but not minimal

A second intuition underlying Theorem I can be explained by means of Figure 5. Suppose that cliques M_w, M_x, M_y and M_z are in the same clique chain. As indicated by the straight arrows, what we would expect this to mean is that clique M_w sends messages to M_x, M_x to M_y, and M_y to M_z. However, when M_x sends messages to M_y, M_y only finds out the thresholds of the players in M_x, but not the thresholds they themselves have observed the players in M_w to have. Yet, M_y knows that M_x relies on the players in M_w having certain thresholds. Therefore, M_y must also get messages from M_w. Similarly, it does not suffice that M_z gets messages from M_y. M_z knows that M_y wants to receives messages from M_x, and by the reasoning above, also from M_w. Therefore, M_z also needs to receive messages from M_x and from M_w.

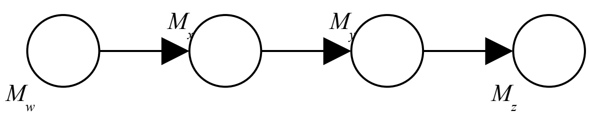


Figure 5. Chain of cliques

While in the example of Figures 1 to 4, there is only a single *msn*, as the next section shows, typically any given threshold game has a plethora of *msn*.

3. Multiplicity of minimal sufficient networks, and existence of endogenous thresholds

Consider the 18-player threshold game $\Gamma_{2,2,3,3,3,4,4,5,6,6,6,7,7,8,9,10,10,10}$, where the numbers denote the players' thresholds. Then an example of a *msn* in the spirit of Chwe (2000) is provided in Figure 6. This example has several characteristics that raise the question whether they are general. *First*, several of the players are in multi-player cliques. The consequence of this is that, starting from a particular number of players who act, the network often cannot add one extra player who acts at a time (when this player happens to be of type *w*). Instead, it is often the case that several players in the same clique all need to be in state *w* to get additional players to act. Put otherwise, a critical mass each time needs to be achieved to get more players to act. *Second*, each individual clique is perfectly homogeneous, in that each clique consists of players with an identical threshold. Indeed, it seems intuitive that similar players would play similar social roles. *Third*, players are at a lower rank in the hierarchy the more conservative they are. It seems intuitive that there would be a one-to-one relationship between the player's threshold and his rank in a chain of cliques. *Fourth*, the minimal sufficient network is connected, in that it connects all players in a single network. *Fifth*, the minimal sufficient network is quite hierarchical, as some players are at rank 4. The example is natural in view of Chwe's interpretation of a *msn* as a hierarchy of social roles.

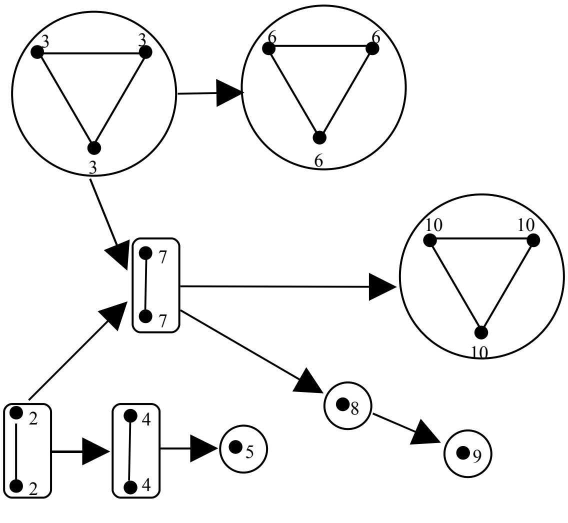


Figure 6. Homogeneous cliques

Yet, as we now point out, this game has many more *msn*s, where each of the characteristics of the example in Figure 6 are violated. *First*, as shown in Figure 7, this game has a *msn* that consists almost exclusively of one-player cliques, and contains players positioned at rank $(t_{\max} - 1) = 9$. In this case, starting from the situation where a number of individual players act, it takes only one extra player in state *w* to get extra players to act. *Second*, this game has minimal sufficient networks containing multi-player cliques that are heterogeneous, in that cliques contain players with different thresholds. An extreme case of this is shown in Figure 8, where the leading clique has a maximal level of heterogeneity, in that players of every possible threshold are represented in it. *Third*, the game has minimal sufficient networks where players with a lower threshold (more radical players) are at a lower in a chain of cliques than players with a higher threshold, meaning that more radical players may only act depending on more conservative players acting. An extreme case of this is shown in Figure 9, where the players in the one-player follower cliques systematically have a threshold lower than or equal to the thresholds of the players in the large leading clique. *Fourth*, minimal sufficient networks exist that consist of three separate, mutually unconnected components, as illustrated in Figure 10.[6] *Fifth*, minimal sufficient networks exist that consist of only two hierarchical ranks, namely a leader rank and a follower rank. This is already illustrated by Figures 8 and 9.

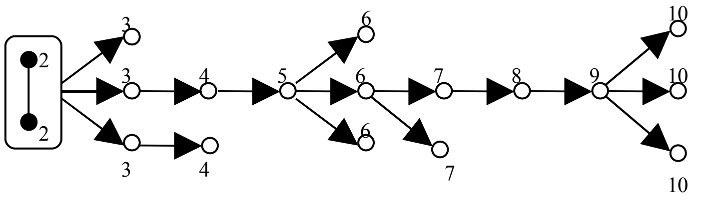


Figure 7 Bandwagon: one-player cliques

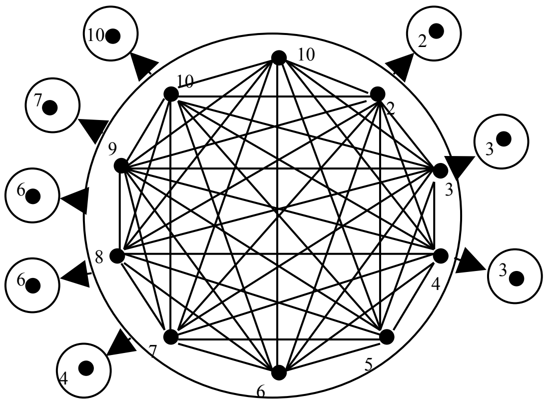
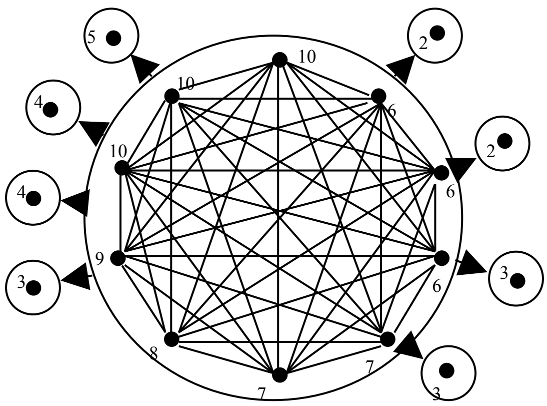


Figure 8 Core-periphery with maximally heterogeneous core



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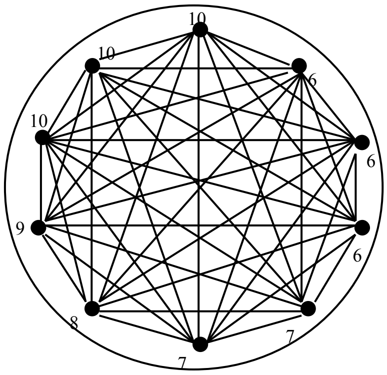
[6] It is clear that each *msn* should contain at least one component containing $t_{\max} = 10$ players. By putting all the highest threshold players in such a 10-player component, one increases the chances that the remaining players are able to achieve collective action among themselves without any connection to the 10-player component. To the rest of the population, one can now apply the same procedure. The maximal threshold in the rest of the population is 5, so that there must be at least one 5-player component, which we again fill up with the most conservative players. One final self-sufficient three-player component can then finally be constructed.

Figure 9 Core-periphery with radical players in periphery



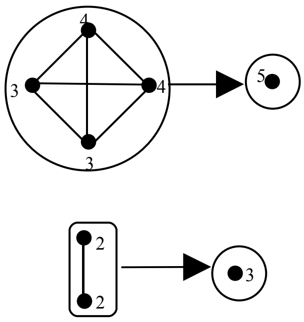


Figure 10 Segregated network

The most surprising elements in these examples are the existence of *msn*s containing heterogeneous cliques, and the existence of *msn*s where players are at a lower rank in a chain of cliques than players who are inherently more conservative than them. Both phenomena can be explained by the fact that a player's exogenously given threshold t_i may be smaller than his *endogenous* threshold determined by his rank and/or social role in the hierarchy. The concept of a player's *endogenous threshold* is formally defined in Definition 7.

Definition 7.
For any minimal sufficient network, define as the *endogenous threshold* $\tilde{t}_x$ of player x the number of players he needs messages from before he acts. Note that $\tilde{t}_x \geq t_x$, and that it may be that $\tilde{t}_x > t_x$.

As an example of a heterogeneous clique, consider the four-player clique with endogenous thresholds (3, 3, 4, 4) in Figure 10. Consider one of the threshold-3 players. Let a wearing a red hat mean that one is of type w (say, available to revolt against the government). Why does our threshold-3 player not content himself with receiving information that two other players of the clique wear a red hat? Suppose that our player would observe this. Then at least one of the two players that he observes wearing a red hat has threshold 4, meaning that this threshold-4 player only acts when seeing three other players wearing a red hat as well. It follows that our threshold-3 player is only assured that at least two players in the clique act when knowing that the threshold-4 player knows that three players in the clique beside him wear a red hat, meaning that the threshold-3 player himself will only act when three players beside himself wear a black hat. Our threshold-3 player thus forms an endogenous threshold of 4.

In fact, the four-player clique in Figure 10 could equally well consist of players with exogenously given thresholds (2, 3, 4, 4). Consider the threshold-2 player in such a clique. Suppose that he would only observe one other player beside himself wearing a red hat. At best, this other player has threshold 3. But then, this threshold-3 player must see two players wearing a red hat. Any such set of two other players that the threshold-3 player will observe will contain at least one threshold-4 player, who must see all players in the clique wearing a red hat. Thus, the threshold-3 player will form endogenous threshold 4. Knowing this, the threshold-2 player will form endogenous threshold 4 as well. Proposition 1 derives a general result for the threshold distribution that any individual clique in a *msn* may have.

Proposition 1. Consider any clique with x players in a minimal sufficient network. Let w be the number of messages that each member of the clique receives from players in other cliques. Consider any integer y, with $1 \le y < x$. Then:

(i) the x-player clique should not contain more than $(y-1)$ players with threshold $(w+y)$ or lower;

(ii) the x-player clique should contain at least two players with threshold $(w+x)$, and no players with a threshold higher than $(w+x)$;

(iii) the x-player clique with *maximal heterogeneity* is one with a single player of threshold y for each integer $y \in [(w+2),(w+x-1)]$, and with two threshold-$(w+x)$ players.

Proof:
(i) If a clique with more than one player contains a threshold-(w+1) player, then this player does not need messages from the other players, and is not in the same clique. If a clique with more than two players contains two players with threshold (w+1) or (w+2), then these two players need not receive message from other players. If a clique with more than three players contains three players with threshold (w+1), (w+2) or (w+3), these players only need messages from one another. Etc.
(ii) By (i), the x-player clique should not contain more than $(x-2)$ players with threshold $(w+x-1)$ or lower. Thus, $(x-1)$ players with threshold $(w+x-1)$ or lower is not allowed. It follows that there should be at least two player with threshold $(w+x)$ or higher. But players with threshold higher than $(w+x)$ do not have enough information to act. It follows that the clique should contain at least two players with threshold $(w+x)$.
(iii) By (i), there can not be any threshold-1 players in the clique. There can be at most one threshold-2 player. If there is a threshold-2 player, there can be at most one threshold-3 player. If there is one threshold-2-player and one threshold-3 player, there can be at most one threshold-4 player. QED

As an extreme example of a player ranked at lower rank than a more conservative player, consider a threshold-2 player in one of the one-player follower cliques in Figure 9. This player, who is inherently one of the most radical in the population is at a lower rank in the population than the most conservative players in the population, namely the threshold-10 players. Because the leading clique is heterogeneous, in fact all players in the leading clique form endogenous threshold 10. In the red hat example, each player in the leading clique only acts when seeing 9 other players in the leading clique wearing a red hat. Because of this fact, observing a single player in the leading clique wearing a red hat does not suffice to our threshold-2 player. Knowing that this player only acts when observing 9 other players in the leading clique wearing a red hat, the threshold-2 player will only act when seeing all 10 players in the leading clique wearing red hats, so that the threshold-2 player forms an endogenous threshold of 11, so that one of our inherently most radical players behaves like a player that is more conservative than one of the inherently most conservative players in the population. In general, such a phenomenon is only possible for players in one-player cliques; also, any followers of our threshold-2 player cannot have lower exogenous threshold than him, since under local knowledge they in any case require messages from the leading clique as well, which suffice to them. These results are formally shown in Proposition 2.

Proposition 2. Consider three cliques directly linked in a single chain, $M_x \rightarrow M_y \rightarrow M_z$. x, y and z denote the number of players in these cliques. Denote by $\tilde{t}_x$ the endogenous threshold of each player in clique M_x. Clique y can contain a player with exogenous threshold $t_y \leq \tilde{t}_x$ if:

(i) $y = 1$;

(ii) the threshold of each player in clique M_z is larger than t_y.

(iii) Denote by w the number of messages received by each player in clique x. Then $t_y < w + 1$.

Proof:

(i) is shown by means of the example in Figure 5. To show (ii), note that a typical player in clique M_x is willing to act because she finds out that at least t_x people are willing to act (which includes herself). The players in M_y receive the same information, but have a lower threshold. It follows that an individual player in M_y does not need any extra messages, and must therefore be in a one-player clique. (iii) follows by a similar reasoning: if the players in M_z have a lower threshold than those in M_y or have the same threshold, they only need messages from M_x, and the messages received by M_x. QED

Proposition 2 should be seen as an exception to a rule: the "natural" order of players according to their thresholds can only be broken for two consecutive ranks, but not for three consecutive ranks. The reason of the reversal in Figure 7 is that there are so few ranks in the first place. General results about the relation between a player's exogenously given threshold and his rank in an *msn* are given in Proposition 3.

Proposition 3. Denote the ranking of cliques in an individual chain by the numbers 1, 2, …, $(z - 2)$, $(z - 1)$, z, where $z \leq (t_{\max} - 1)$. The following rules apply for the manner in which cliques are ordered along individual chains.

(i) In any individual chain, a threshold-2 player can be located at rank 1 or 2. A player with exogenously given threshold $x_i > 2$ is located at rank r, with $r \leq (x_i - 1)$;

(ii) In any individual chain with highest rank z, a player with threshold $(t_{\max} - y)$ is located at rank r, with $r \geq (z - y - 1)$.

Proof:

(i) As each player receives messages from all players higher up in the hierarchy, and as each clique contains at least one player, a player i at rank r with threshold x_i receives messages from at least $(r-1)$ players at rank ρ such that $\rho \leq (r-1)$, from at least $(r-2)$ players at rank ρ such that $\rho \leq (r-2)$, from at least $(r-3)$ players at position ρ such that $\rho \leq (r-3)$, etc. If $x_i \leq (r-2)$, $x_i \leq (r-3)$, etc., then player i does not need messages from the players at ranking r, $(r-1)$, etc.

(ii) A player with threshold t_{max} cannot be located at ranking x with $x \leq (z-2)$: otherwise, the players at ranking z do not need links to the players at ranking $(z-1)$. A player with threshold $(t_{\max} - 1)$ cannot be located at ranking x with $x \leq (z-3)$: otherwise, the players at ranking $(z-1)$ do not need links to the players at ranking $(z-2)$. Etc.

QED

The existence of *msn* with endogenous thresholds does not contradict Chwe's interpretation of hierarchies of cliques as hierarchies of social roles. The only correction is that a player's social role may be quite different from his endogenous threshold. A player with leader capacities (= a low threshold) may be a follower, a player with few inherent capacities for leadership may be a leader. Very diverse players may play one and the same role, and very similar people may play very different roles. This is because of the discrepancy between the players' inherent tendencies, and the behaviour forced upon them by their network environment. Thus, our model allows for the effect of embeddedness in a network on a player's behavior (Granovetter, 1985). It is not only the case that players' individual behavioral tendencies determine how they are positioned in a social hierarchy; the social hierarchy itself also determines how they behave.

Why are there so many *msn*? We now show some general results. A first insight that can be obtained that each *msn* of a particular *n*-player threshold game with maximal threshold t_{max} must contain at least one component connecting exactly t_{max} players. For instance, all *msn* in Figure 6 to 10 contain a ten-player component.

Lemma 4. Consider any *n*-player threshold game with maximal threshold t_{max}. Then any *msn* contains at least one component connecting exactly t_{max} players.
Proof: Any t_{max}-threshold player must receive messages from at least (t_{max} – 1) different players. It follows that any *msn* must contain at least one component of t_{max} connected players.

A second insight is that, for any subset of players containing at least one t_{max}-threshold player, one can find a minimal sufficient component.

Lemma 5. For any subset of t_{max} players containing at least one t_{max}-threshold player, a connected minimal sufficient component exists.
Proof: For any set of t_{max} players containing at least one t_{max}-threshold player, one can construct a connected minimal sufficient component by the following network formation algorithm. In Step 1, build as many leading cliques of size 2 as possible (containing threshold (2, 2)). Next, among the remaining players, build as many leading cliques of size 3 as possible ((3, 3, 3) or (3, 3, 2)). After that, build among the remaining players as many leading cliques of size 4 as possible ((4, 4, 4, 4) or (4, 4, 4, 3), (4, 4, 3, 3), or (4, 4, 3, 2), etc. Continue this procedure until it is not possible anymore to build leading cliques. Note that application of Step 1 may lead all t_{max} players to be put in a single clique. In Step 2, if there are any remaining players, connect as many of them as possible in follower cliques of size 1 to one or more of the leading cliques (a threshold-2 player connected to a two-player leading clique, a threshold-3 player connected to a two- or three-player leading clique; a threshold-4 player to one three-player leading clique or two two-player leading cliques, etc.). After this, among the remaining players connect as many follower cliques of size 2 to one or more of the leading cliques. Continue this procedure until it is no longer possible to connect players to the leading cliques. In Step 3, if any players remain, connect as many as possible one-player cliques to one or more of the follower cliques. Next two-player cliques. Etc. Continue these steps until all players have been allocated. Note that this procedure necessarily results in a connected component, as the t_{max} player needs to receive messages from all other players.

A third insight is that one can always connect the players not included in any subset as defined in Lemma 5 to a minimal sufficient component defined in Lemma 5.

Lemma 6. Consider any t_{max}-player connected minimal sufficient component containing at least one t_{max}-threshold player. Then any remaining players in the population can be connected to this component in one-player follower cliques.

Proof: Any minimal sufficient component as defined in Lemma 5 consists of a partition of ten players in cliques according to endogenously-formed thresholds. Denote the set of endogenous thresholds contained as (x_1, x_2,…, x_i,…, x_z). For any player not included in the minimal sufficient component with exogenously given threshold t_j, look for the smallest x_i in the set (x_1, x_2,…, x_i,…, x_z) such that $t_j \geq (x_i - 1)$. Then the t_j-threshold player can be connected in a one-player follower clique to this clique of the minimal sufficient component with endogenous threshold x_i. QED

In order to see the plethora of *msn*s, note first that there is a plethora of subsets containing at least one threshold-10 player that can be formed in game $\Gamma_{2,2,3,3,3,4,4,5,6,6,6,7,7,8,9,10,10,10}$. By Lemmata 4 to 6, one can form a different *msn* for each of these subsets, by following the procedure set out in Lemmata 5 to 6. Yet, often one can even form other minimal sufficient components for any subset of 10 players containing at least one threshold-10 players. And there are often several ways to connect the remaining players to such a minimal sufficient component. Moreover, as illustrated in Figure 8, cases exist where one can form isolated components with the remaining players.

Concluding this section, there typically is a plethora of *msn*, causing the players a large amount of strategic uncertainty. Yet, if we see players as repeatedly being involved in threshold games, where the population may each time differ, then this plethora of equilibria can be seen as an opportunity rather than as a threat. As shown by Granovetter (1978), some *msn* stop being minimal sufficient even for small changes in the population. Yet, because of the plethora of *msn* for any typical game, perhaps there are other *msn*, or at least *msn* with certain characteristics, that work no matter how the threshold in the population changes, i.e. that work for all populations. Such go-for-all *msn* may then be focal, and the problem of strategic uncertainty is then resolved. The next section investigates whether such generally applicable network formation rules exist.

4. Generalizability of network formation rules

The purpose of this section is to investigate whether we can find, across all allowable populations of players of our game, *msn* that have a certain common feature. If such *msn* with a common feature exist, this common feature may become focal and solve the strategic uncertainty arising from the multiplicity of *msn*. The common feature itself may then be seen as a rule for successful network formation. We consecutively consider the following network formation rules. A first rule we treat (Section 3.1) corresponds to Granovetter's bandwagons (1978), and consists of putting as many players as possible in one-player cliques, and of give each player a rank corresponding to his order in the ranking of exogenously given thresholds. A second rule (Section 3.2) treated corresponds to one of the main examples provided by Chwe (2000) and consists of putting all players with the same threshold together in one and the same homogeneous clique. A third rule (Section 3.3) consists of constructing a core-periphery network, with a number of players in the leading clique equal to the largest threshold in the population, and the rest of the players in one-player follower cliques of this leading clique.

4.1 Bandwagons

Definition 7.
Define as a *bandwagon* a network where all players are in one-player cliques, with the exception of two threshold-2 players, who are in a two-player leading clique.

In sociological terms, a bandwagon is almost exclusively characterized by weak links between players (Granovetter, 1973). With the exception of the leading clique, there are no multi-player cliques in which strong links are formed.

Proposition 4 shows that a bandwagon exists as long as there are at least two threshold-2 players, and as long as for any player with threshold x, there are at least $(x - 1)$ players with a threshold smaller than him. Put otherwise, if there is a player with a threshold x higher than 2, there must be at least $(x - 1)$ more radical players who act without receiving information from this player. Put otherwise, a bandwagon exists if any subset of 2, 3, 4,..., $(t_{\max} - 1)$ players can be made to act independently. The population is therefore relatively radical, as the most radical players act without little assurance that others act. As the most radical players have the leadership, in that nobody acts unless they act, they must also be self-sufficient.

Proposition 4. A bandwagon exists if there are at least two threshold-2 players, and for each threshold-x player with $x > 2$ at least $(x - 1)$ players with threshold lower than x.
Proof:
If there are less than two threshold-2 players, any *msn* automatically contains a leading clique with at least three players. Furthermore, in order for any threshold-3 player not to require being in a multi-player clique, there must be at least two threshold-2 players from which he can receive individual messages. For any threshold-4 not to require being in a multi-player clique, there must be at least three players with threshold 3 or lower from which he can receive individual messages. Etc. QED

The simplest case of a bandwagon is where there are no gaps between the players thresholds, as depicted in the example in Figure 5. Note that if there are more than two threshold-2 players, the remainder can still be linked in one-player cliques to the leading clique of two threshold-2 players; these followers than form endogenous threshold 3. It should also be noted that a bandwagon exists even if there are gaps between the thresholds, in which case the bandwagon contains "loops" and no longer has a tree structure.

A bandwagon may be considered as a simple network formation rule. First, order all players according to their thresholds. Then, let each player receive the necessary number of messages from more radical players. Unfortunately, bandwagons do not exist for all populations. We illustrate this by means of two 10-player games, which are exceptional in that only a single *msn* may exist. For instance, in game $\Gamma_{2,2,3,4,5,6,7,8,9,10}$ the bandwagon is the only msn. But game $\Gamma_{2,3,4,5,6,7,8,9,10,10}$, a bandwagon does not exist, and the only *msn* is the antipode of the bandwagon, consisting of a single clique containing all players.

4.2 Homogeneous cliques

Definition 8.
Define as a homogeneous *msn* where all players with the same threshold are in one and the same clique.

The idea is that all players of the same type play exactly the same social role. This may be seen as reflecting the often made observation of homophily in networks observed in sociology: similar players tend to link to one another (see McPherson, Smith Lovin and Cook, 2001). The similar players form strong links with one another, and are connected with dissimilar players only through weak links.

Proposition 5. A homogeneous *msn* exists if for each y players with threshold x

(i) for players in any leading clique, it is the case that $y = x$.

(ii) for players in a follower clique, it is the case that $y < (x - 1)$.

(iii) for players in a follower clique, there must be exactly $(x - y)$ players with thresholds lower than x who can be put in homogeneous cliques.

Proof:

(i) A leading clique acts without receipt of any incoming messages. As all players must be in homogeneous cliques, it any leading clique it must be the case that $y = x$.

(ii) Threshold-x players can be in a clique of at most x players. It there are more than x players, it is not possible for all of them to be in the same clique. If there are $(x - 1)$ threshold-x players in a follower clique, then these players must receive information from a single player. By Corollary 1, this player must be in a one-player clique. As each clique receives messages from all cliques higher up in a chain, this one-player clique can itself not be a follower clique. But not player in the considered populations can be in a one-player leading clique.

(iii) If there are less than $y < x$ players with threshold x, then the threshold-x players only act when receiving information from exactly $(x - y)$ players, who must themselves be in homogeneous cliques.

Homogeneous *msn* apply only to particular populations. As any *msn* has at least one leading clique, there must be at least one threshold level x for which there are exactly x players. Moreover, the number of players of a certain threshold puts restrictions on the rest of the population. For instance, if there are four threshold-6 players, there must be exactly two threshold-2 players. If there are three threshold-6 players, then either there must be exactly three threshold-3 players, or exactly two threshold-2 players and exactly one threshold-3 player. If there are two threshold-6 players, then there must be exactly four threshold-4 players, or exactly two threshold-2 players and exactly two threshold-4 players. For the exceptional case where there exactly ten players, one can find examples where the only *msn* is homogeneous, e.g. $\Gamma_{2,2,4,4,6,6,8,8,10,10}$. Yet, in game $\Gamma_{2,3,4,5,6,7,8,9,10,10}$, the only *msn* contains a single perfectly heterogeneous multi-player clique containing players of all thresholds. Game $\Gamma_{2,2,4,4,6,6,8,8,10,10}$ conveniently has gaps between the players' thresholds, but such gaps are not a necessary condition for the existence of a homogenous *msn*. As illustrated in Figure 4, by having multiple chains of cliques in a *msn*, a homogenous msn is possible even without gaps between the thresholds.

4.3 Core-periphery

After studying two network formation rules inspired on examples from the literature, we now turn to a third network formation rule. We have already noted in Lemma 5 that any network contains at least one minimal sufficient component consisting of exactly $t_{\max}$ players and containing at least one $t_{\max}$-threshold player. By Lemma 6, a *msn* can always be found by constructing a single such $t_{\max}$-player minimal sufficient component, and connecting any remaining players to it in one-player follower cliques. The problem now is that, for any subset of $t_{\max}$ players, many minimal sufficient components may exist, again inducing different ways to link the remaining players in one-player cliques, such that the problem of multiplicity is not solved. This is why we study the case where the $t_{\max}$-player minimal sufficient component takes a particular form, in being complete.

Definition 9.
Define as a core-periphery *msn* any *msn* with $t_{\max}$ players in a single complete leading clique, and all other players in one-player follower cliques around it.

In a core-periphery network, we have a single leading clique with strong links, which has weak links with individual players.

Proposition 7.
The following condition is necessary and sufficient for the existence core-periphery *msn*. Let there exist at least one subset N' with $N' \subseteq N$ consisting of a number of $t_{\max}$ players ($i, j, \ldots, w, y, z$), ordered in ascending order according to the levels of their thresholds, and with the following characteristics. Player *i* has threshold $t_i \geq 2$, player *j* has threshold $t_j \geq 3, \ldots$, player *w* has threshold $t_w \geq (t_{\max} - 1)$, and players *y* and *z* both have thresholds $t_{\max}$.
Proof:
This result follows directly from Propositions 1 and 3.

Again, sticking to the exceptional case of games where there are exactly $t_{\max}$ players, it is easy to construct examples where core-periphery is not an *msn*, such as $\Gamma_{2,2,4,4,6,6,8,8,10,10}$ (only homogeneous cliques) and $\Gamma_{2,2,3,4,5,6,7,8,9,10}$ (only bandwagon). Thus, unfortunately, the core-periphery is not generally applicable either. Whereas the bandwagon applies to relatively radical populations, where instigators need few messages from other players, one can say that core-periphery applies to relative conservative populations. For instance, if there is no option but to fill any core with at least two threshold-2 players, then the core-periphery is not a *msn*. The leading clique must contain at least a few of the most conservative players; these infect the rest of the population, making their endogenous thresholds equal to the exogenous threshold of the most conservative players. Collective action only takes place if everybody agrees that collective action is worth a while.

Rounding up this section, we conclude that none of the treated network formation rules allow for the formation of a similar *msn* for each possible threshold game, and a solution to the problem of strategic uncertainty is thus not found. Yet there are important differences between the several network formation rules treated. The reason why a bandwagon or a homogenous clique network are not minimal sufficient for all possible populations is that they are not sufficient for all populations. The existence of a bandwagon *msn* crucially relies on there being at least two threshold-2 players. While the bandwagon is sufficient for game $\Gamma_{2,2,3,4,5,6,7,8,9,10}$, it is not for game $\Gamma_{2,3,4,5,6,7,8,9,10,10}$. In our main example $\Gamma_{2,2,3,3,3,4,4,5,6,6,6,7,7,8,9,10,10,10}$, if one of the threshold-3 player's threshold is turned into 4, a *msn* where all players with the same thresholds are in the same homogenous cliques, so that the three threshold-4 players are in a leading clique, is no longer sufficient.

The reason why the core-periphery network is not always a *msn*, however, is because it is not always minimal. Yet, the core-periphery network is always sufficient. Any player positioned anywhere in a core-periphery always knows that nine other players are willing to act. This fact yield an additional advantage to core-periphery networks. If players are willing to compromise on the fact that core-periphery networks are not always minimal (even they are minimal as well quite often), then it does not matter who is where in the network. In order to achieve the bandwagon, or a network with homogenous cliques, players need to coordinate on who is positioned where in the network. Core-periphery networks on the contrary are sufficient even if players are put at random places in the network. The next section formalizes the idea of unsophisticated players in treating a modified game where players additionally do not know each other's thresholds.

6. Players do not know each other's thresholds

In this section, we treat a modified game where players only know that they are playing an n-player threshold game where any player's maximal threshold is t_{max}, and their own threshold. In particular, for each n and t_{max}, there is a distribution of possible populations, and this distribution is common knowledge. The populations that occur with positive probability include a population where all other players have threshold t_{max}. We continue to assume that, separately from their thresholds, players may be either willing or unwilling. Otherwise, a Nah equilibrium exists would exist where players take collective action even in the absence of communication. We continue to assume that a link $g_{i,j} = 1$ enables agent j to access i's information on whether i is in state w or x. We assume that such a link does not allow player j to find out player i's threshold.

It is easy to see now that for large enough L, the core-periphery network is a *msn* in this modified game. To see why, note that it continues to be the case that each *msn* must contain at least one component consisting of t_{max} players. Moreover, given each individual player's uncertainty about the other players' thresholds, this component can only be complete. Each individual player in this component, even when having only threshold 2, considers the possibility that all other players have threshold t_{max}, and develops endogenous threshold t_{max}. The players not included in the leading clique similarly develop endogenous thresholds of t_{max}. They only need messages from the players in the leading clique, so that they are automatically in one-player cliques. An exception is the case where $n \geq 2t_{max}$, in which case multiple complete components can be formed. However, this is not possible for populations where $n < 2t_{max}$. Thus, the unique *msn* consists of putting a random sample of t_{max} players in a complete leading clique (core), and to put all other players in one-player follower cliques of this leading clique (periphery). The mechanism by which the core-periphery network is generally applicable is that it changes each player's endogenous threshold in such a way that he acts like the most conservative player in the game. Typically, the leading clique is heterogeneous. The standard argument for heterophily in networks (e.g. Reagans and Zuckerman, 2001) is that heterophily in networks promotes innovation, because of a wider range of opinions. In the present model, heterophily is called for because it is leads to networks that are generally applicable.

7. Conclusion

The proposed core-periphery network as a general recipe to achieve collective action in the threshold game has some intuitive appeal. First, a representative committee is randomly selected from the population. Indeed, on average, the committee is distributed in the same manner as the entire population. Also, the committee is large enough to convince even the most conservative player to act. Thus, the purpose of the committee is to achieve consensus about the desirability of collective action. Second, once the representative committee has achieved consensus that collective action is desirable, this is communicated to each other member of the population individually. Given that the committee is large enough, the other players will be convinced to take action, whatever their degree of conservatism.

Several issues remain to be investigated. *First*, we have neglected the cost of forming links. Because of size of the leading clique and the fact that it is complete, many links are needed in the core-periphery network, and the bandwagon is more economic in this sense (even though it needs to be stressed that each player still needs to receive a message from each player higher up in a clique chain as well). *Second*, one could envisage noise in the links, where with small probability information a player's information does not get through to others. In this case, the players make themselves very vulnerable in the core-periphery. The information of a single player in the core not getting through suffices to disable all collective action. The more players there are in the core, the more likely this is to happen. In a bandwagon, however, the probability that messages get lost in the smaller leading clique is relatively small, so that it is likely that at least some players get to act. *Third*, it may be that a strategic player tries to disrupt collective action, e.g. by eliminating one player and doing this with the maximal possible disruptive effect. Given the examples of strikes, revolutions and riots, this is plausible. Consider for instance the game $\Gamma_{2,2,2,2,2,2,2,2,2,2}$. In the core-periphery network, one leading clique of two players is formed, with the eight remaining players around it in one-player cliques. An alternative (but not generally applicable) segregated network has five segregated components of two players. Consider a strategic adversary of the network whose purpose it is to disrupt the network, and who can do this by eliminating any one player. In the core-periphery network, by eliminating one of the players in the leading clique, the disruptor can make sure that no player acts. In the segregated network, the number of acting players can only be reduced to eight. If some players acting is better than no players acting, the segregated network, consisting of segregated "cells", is better.

References

Aumann, R. (1990) Nash equilibria are not self-enforcing, in: *Economic Decision Making: Games, Econometrics and Optimisation*. J. Gabszewicz, J.-F. Richard, and L. Wolsey (eds.), Elsevier, 201-206.

Bala, V. & Goyal, S. (1998) Learning from neighbours, *Review of Economic Studies* 65, 595-621.

Carlsson, H. and Damme, E. van (1993) Equilibrium selection in stag hunt games, in: *Frontiers of Game Theory*, K. Binmore, A. Kirman and P. Tanki (eds.), MIT Press.

Chwe, M. S.-Y. (1999) Structure and strategy in collective action, *American Journal of Sociology* 105, 128-156.

Chwe, M. S.-Y. (2001) Communication and coordination in social networks, *Review of Economic Studies* 67, 1-16.

Cooper, R., DeJong, D.V., Forsythe, R. and Ross, T.W. (1992) Communication in coordination games, *The Quarterly Journal of Economics* 107, 739-771.

Corbae, D. and Duffy, J. (2008) Experiments with network formation, *Games and Economic Behavior* 64, 81-120.

Granovetter, M. (1973) The strength of weak ties, *American Journal of Sociology* 78, 1360-1380.

Granovetter, M. (1978) Threshold models of collective behaviour, *American Journal of Sociology* 83, 1420-1443.

Granovetter, M. (1985) Economic action and social structure: the problem of embeddedness, *American Journal of Sociology* 91, 481-510.

Harsanyi, J. and Selten, R. (1988) *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge Ma.

Jackson, M.O. A survey of models of network formation: stability and efficiency, *working paper*, California Institute of Technology.

Jackson, M.O. & Wolinsky, A. (1996) A strategic model of social and economic networks, *Journal of Economic Theory* 71, 44-74.

Lazer, D. (2001) The co-evolution of individual and network, *Journal of Mathematical Sociology* 25, 69-108.

McPherson, M., Smith-Lovin, L. and Cook, J.M. (2001) Birds of a feather: homophily in social networks, *Annual Review of Sociology* 27, 415-444.

Reagans, R. & Zuckerman, E.W. (2001) Networks, diversity and productivity – the social capital of corporate R&D teams, *Organization Science* 12, 502-517.

Rubinstein, A. (1989) The electronic mail game – strategic behavior under 'almost common knowledge', *American Economic Review* 79, 385-391.

Skyrms, B. (2004) *The stag hunt and the evolution of social structure*, Cambridge University Press, Cambridge UK.

Vanderschraaf, P. (2008) Game theory meets threshold analysis – reappraising the paradoxes of anarchy and revolution, *British Journal for the Philosophy of Science* 59, 579-617.